Interval valued $(\in, \in \vee q)$-fuzzy filters of pseudo BL-algebras

Jianming Zhan[a,*], Wiesław A. Dudek[b], Young Bae Jun[c]

[a] Department of Mathematics, Hubei Institute for Nationalities,
Enshi, Hubei Province, 445000, P. R. China

[b] Institute of Mathematics and Computer Science, Wrocław University of Technology,
Wybrzeże Wyspiańskiego 27, 50-370 Wrocław, Poland

[c] Department of Mathematics Education, Gyeongsang National University,
Chinju 660-701, Korea

Abstract

We introduce the concept of quasi-coincidence of a fuzzy interval value with an interval valued fuzzy set. By using this new idea, we introduce the notions of interval valued $(\in, \in \vee q)$-fuzzy filters of pseudo BL-algebras and investigate some of their related properties. Some characterization theorems of these generalized interval valued fuzzy filters are derived. The relationship among these generalized interval valued fuzzy filters of pseudo BL-algebras is considered. Finally, we consider the concept of implication-based interval valued fuzzy implicative filters of pseudo BL-algebras, in particular, the implication operators in Lukasiewicz system of continuous-valued logic are discussed.

Keywords: Pseudo BL-algebra; filter; interval valued $(\in, \in \vee q)$-fuzzy filter; fuzzy logic; implication operator.

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* Corresponding author.
E-mail address: zhanjianming@hotmail.com (J. Zhan), dudek@im.pwr.wroc.pl (W. A. Dudek)

1 Introduction

Logic appears in a "scared" form (resp., a "profane") which is dominant in proof theory (resp., model theory). The role of logic in mathematics and computer science is twofold-as a tool for applications in both areas, and a technique for laying the foundations. Non-classical logic including many-valued logic, fuzzy logic, etc., takes the advantage of the classical logic to handle information with various facets of uncertainty (see [29] for generalized theory of uncertainty), such as fuzziness, randomness, and so on. Non-classical logic has become a formal and useful tool for computer science to deal with fuzzy information and uncertain information. Among all kinds of uncertainties, incomparability is an important one which can be encountered in our life.

BL-algebras were introduced by Hájek as algebraic structures for his Basic Logic, starting from continuous t-norm and their residuals ([18]). MV-algebras [4], product algebras and Gödel algebras are the most classes of BL-algebras. Filters theory play an important role in studying these algebras. From logical point of view, various filters correspond to various sets of provable formulae. Hájek [18] introduced the concepts of (prime) filters of BL-algebras. Using prime filters of BL-algebras, he proved the completeness of Basic Logic BL. BL-algebras are further discussed by Di Nola ([9] and [10]), Iorgulescu ([19]), Ma([20]) and Turunen ([24], [25]), and so on.

Recent investigations are concerned with non-commutative generalizations for these structures (see [8, 11 - 17, 22 - 25, 32 - 33]). In [16], Georgescu et al. introduced the concept of pseudo MV-algebras as a non-commutative generalization of MV-algebras. Several researchers discussed the properties of pseudo MV-algebras(see [11], [12], [22] and [23]). Pseudo BL-algebras are a common extension of BL-algebras and pseudo MV-algebras (see [8], [13], [14], [17], [32]). These structures seem to be a very general algebraic concept in order to express the non-commutative reasoning. We remark that a pseudo BL-algebra has two implications and two negations.

After the introduction of fuzzy sets by Zadeh [27], there have been a number of generalizations of this fundamental concept. In [28], Zadeh made an extension of the concept of a fuzzy set (i.e., a fuzzy set with an interval valued membership function). The interval valued fuzzy subgroups were first defined and studied by Biswas [3] which are the subgroups of the same natural of the fuzzy subgroups defined by Rosenfeld. A new type of fuzzy subgroup, that is, the $(\in, \in \vee q)$-fuzzy subgroup, was introduced in an earlier paper of Bhakat and Das [2] by using the combined notions of "belongingness" and "quasicoincidence" of fuzzy points and fuzzy sets, which was introduced by Pu and Liu [21]. In fact, the $(\in, \in \vee q)$-fuzzy subgroup is an important generalization of Rosenfeld's fuzzy subgroup. Recently, Davvaz [5] applied this theory to near-rings and obtained some useful results. Further, Davvaz and Corsini [6] redefined fuzzy H_v-submodule and

many valued implications. In [31], Zhan et al. also discussed the properties of interval valued $(\in, \in\vee q)$-fuzzy hyperideals in hypernear-rings. For more details, the reader is referred to [5], [6] and [31].

The paper is organized as follows. In section 2, we recall some basic definitions and results of pseudo BL-algebras. In section 3, we introduce the notion of interval valued $(\in, \in\vee q)$-fuzzy filters in pseudo BL-algebras and investigate some of their related properties. Further, the notions of interval valued $(\in, \in\vee q)$-fuzzy (implicative, MV- and G-) of pseudo BL-algebras are introduced and the relationship among these generalized interval valued fuzzy filters of pseudo BL-algebras is considered in section 4. Finally, in section 5 we consider the concept of implication-based interval valued fuzzy implicative filters of pseudo BL-algebras, in particular, the implication operators in Lukasiewicz system of continuous-valued logic are discussed.

2 Preliminaries

A pseudo BL-algebra is an algebra $(A; \wedge, \vee, \odot, \to, \hookrightarrow, 0, 1)$ of type (2,2,2,2,2, 0,0) such that $(A, \wedge, \vee, 0, 1)$ is a bounded lattice, $(A, \odot, 1)$ is a monoid and the following axioms

(a_1) $x \odot y \leq z \Longleftrightarrow x \leq y \to z \Longleftrightarrow y \leq x \hookrightarrow z$;

(a_2) $x \wedge y = (x \to y) \to x = x \odot (x \hookrightarrow y)$;

(a_3) $(x \to y) \vee (y \to x) = (x \hookrightarrow y) \vee (y \hookrightarrow x) = 1$

are satisfied for all $x, y, z \in A$.

We assume that the operations $\vee$, $\wedge$, $\odot$ have priority towards the operations $\to$ and $\hookrightarrow$.

Example 2.1. (Di Nola [8]) Let $(G, \vee, \wedge, +, -, 0)$ be an arbitrary l-group and let θ be the symbol distinct from the element of G. If $G^- = \{x' \in G | x' \leq 0\}$, then we define on $G^* = \{\theta\} \cup G^-$ the following operations:

$$x' \odot y' = \begin{cases} x' + y' & \text{if} \quad x', y' \in G^-, \\ \theta & \text{otherwsie}, \end{cases}$$

$$x' \to y' = \begin{cases} (y' - x') \wedge 0 & \text{if} \quad x', y' \in G^-, \\ \theta & \text{if} \quad x' \in G^-,\ y' = \theta, \\ 0 & \text{if} \quad x' = \theta, \end{cases}$$

$$x' \hookrightarrow y' = \begin{cases} (-x' + y') \wedge 0 & \text{if} \quad x', y' \in G^-, \\ \theta & \text{if} \quad x' \in G^-,\ y' = \theta, \\ 0 & \text{if} \quad x' = \theta. \end{cases}$$

If we put $\theta \leq x'$, for any $x' \in G$, then $(G^*, \leq)$ becomes a lattice with first element θ, and the last element 0. The structure $G^* = (G^*, \vee, \wedge, \odot, \rightarrow, \hookrightarrow, 0 = \theta, 1 = 0)$ is a pseudo BL-algebra.

Let A be a pseudo BL-algebra and $x, y, z \in A$. The following statements are true (for details see [8, 15, 32]):

(1) $(x \odot y) \rightarrow z = x \rightarrow (y \rightarrow z)$,

(2) $(y \odot x) \hookrightarrow z = x \hookrightarrow (y \hookrightarrow z)$,

(3) $x \leq y \Longleftrightarrow x \rightarrow y = 1 \Longleftrightarrow x \hookrightarrow y = 1$,

(4) $(x \hookrightarrow y) \hookrightarrow x \leq (x \hookrightarrow y) \rightarrow ((x \hookrightarrow y) \hookrightarrow y)$,

(5) $x \leq y \Rightarrow x \odot z \leq y \odot z$,

(6) $x \leq y \Rightarrow z \odot x \leq z \odot y$,

(7) $x \odot y \leq x, \;\; x \odot y \leq y$,

(8) $x \odot 0 = 0 \odot x = 0$,

(9) $1 \rightarrow x = 1 \hookrightarrow x = x$,

(10) $y \leq x \rightarrow y$.

A non-empty subset I of a pseudo BL-algebra A is called a *filter* of A if it satisfies the following two conditions:
(i) $x \odot y \in I$ for all $x, y \in I$;
(ii) $x \leq y \Longrightarrow y \in I$ for all $x \in I$ and $y \in A$.

Remind that a filter I is called
implicative if $\begin{cases} (x \rightarrow y) \hookrightarrow x \in I \text{ implies } x \in I, \\ (x \hookrightarrow y) \rightarrow x \in I \text{ implies } x \in I, \end{cases}$
MV-filter if $\begin{cases} x \rightarrow y \in I \text{ implies } ((y \rightarrow x) \hookrightarrow x) \rightarrow y \in I, \\ x \hookrightarrow y \in I \text{ implies } ((y \hookrightarrow x) \rightarrow x) \hookrightarrow y \in I, \end{cases}$
G-filter if $\begin{cases} x \rightarrow (x \rightarrow y) \in I \text{ implies } x \rightarrow y \in I, \\ x \hookrightarrow (x \hookrightarrow y) \in I \text{ implies } x \hookrightarrow y \in I. \end{cases}$

Now, we introduce the concept of fuzzy (implicative) filters of pseudo BL-algebras as follows:

Definition 2.2. A fuzzy set μ of a pseudo BL-algebra A is called a *fuzzy filter* of A if
(i) $\mu(x \odot y) \geq \min\{\mu(x), \mu(y)\}$,
(ii) $x \leq y \Longrightarrow \mu(x) \leq \mu(y)$,
is satisfied for all $x, y \in A$.

Definition 2.3. A fuzzy filter μ of a pseudo BL-algebra A is called a *fuzzy implicative filter* if

(iii) $\mu(x) \geq \max\{\mu((x \to y) \hookrightarrow x), \mu((x \hookrightarrow y) \to x),$

holds for all $x, y, z \in A$.

For any fuzzy set μ of A and $t \in (0,1]$, the set $\mu_t = \{x \in A \,|\, \mu(x) \geq t\}$ is called a *level subset* of μ.

It is not difficult to verify that the following theorem is true.

Theorem 2.4. *A fuzzy set μ of a pseudo BL-algebra A is a fuzzy filter of A if and only if each its non-empty level subset is a filter of A.* $\square$

By an *interval number* $\widehat{a}$, we mean an interval $[a^{\perp}, a^{\top}]$, where $0 \leq a^{\perp} \leq a^{\top} \leq 1$. The set of all interval numbers is denoted by $D[0,1]$. The interval $[a,a]$ can be simply identified with the number $a \in [0,1]$.

For the interval numbers $\widehat{a}_i = [a_i^{\perp}, a_i^{\top}]$, $\widehat{b}_i = [b_i^{\perp}, b_i^{\top}] \in D[0,1]$, $i \in I$, we define

$$\mathrm{rmax}\{\widehat{a}_i, \widehat{b}_i\} = [\max\{a_i^{\perp}, b_i^{\perp}\}, \max\{a_i^{\top}, b_i^{\top}\}],$$
$$\mathrm{rmin}\{\widehat{a}_i, \widehat{b}_i\} = [\min\{a_i^{\perp}, b_i^{\perp}\}, \min\{a_i^{\top}, b_i^{\top}\}],$$
$$\mathrm{rinf}\, \widehat{a}_i = [\bigwedge_{i \in I} a_i^{\perp}, \bigwedge_{i \in I} a_i^{\top}], \qquad \mathrm{rsup}\, \widehat{a}_i = [\bigvee_{i \in I} a_i^{\perp}, \bigvee_{i \in I} a_i^{\top}]$$

and put

(1) $\widehat{a}_1 \leq \widehat{a}_2 \Longleftrightarrow a_1^{\perp} \leq a_2^{\perp}$ and $a_1^{\top} \leq a_2^{\top}$,

(2) $\widehat{a}_1 = \widehat{a}_2 \Longleftrightarrow a_1^{\perp} = a_2^{\perp}$ and $a_1^{\top} = a_2^{\top}$,

(3) $\widehat{a}_1 < \widehat{a}_2 \Longleftrightarrow \widehat{a}_1 \leq \widehat{a}_2$ and $\widehat{a}_1 \neq \widehat{a}_2$,

(4) $k\widehat{a} = [ka^{\perp}, ka^{\top}]$, whenever $0 \leq k \leq 1$.

Then, it is clear that $(D[0,1], \leq, \vee, \wedge)$ is a complete lattice with $0 = [0,0]$ as its least element and $1 = [1,1]$ as its greatest element.

The interval valued fuzzy sets provide a more adequate description of uncertainty than the traditional fuzzy sets; it is therefore important to use interval valued fuzzy sets in applications. One of the main applications of fuzzy sets is fuzzy control, and one of the most computationally intensive part of fuzzy control is the "defuzzification". Since a transition to interval valued fuzzy sets usually increase the amount of computations, it is vitally important to design faster algorithms for the corresponding defuzzification. For more details, the reader can find some good examples in [7] and [30].

Recall that an *interval valued fuzzy set* F on X is the set

$$F = \{(x, [\mu_F^{\perp}(x), \mu_F^{\top}(x)]) \,|\, x \in X\},$$

where $\mu_F^{\perp}$ and $\mu_F^{\top}$ are two fuzzy subsets of X such that $\mu_F^{\perp}(x) \leq \mu_F^{\top}(x)$ for all $x \in X$. Putting $\widehat{\mu_F}(x) = [\mu_F^{\perp}(x), \mu_F^{\top}(x)]$, we see that $F = \{(x, \widehat{\mu_F}(x)) \,|\, x \in X\}$, where $\widehat{\mu_F} : X \to D[0,1]$.

If A, B are two interval valued fuzzy sets of X, then we define

$A \subseteq B$ if and only if for all $x \in X$, $\mu_A^{\perp}(x) \leq \mu_B^{\perp}(x)$ and $\mu_A^{\top}(x) \leq \mu_B^{\top}(x)$,

$A = B$ if and only if for all $x \in X$, $\mu_A^{\perp}(x) = \mu_B^{\perp}(x)$ and $\mu_A^{\top}(x) = \mu_B^{\top}(x)$.

Also, the union, intersection and complement are defined as follows:

$A \cup B = \{(x, [\max\{\mu_A^{\perp}(x), \mu_B^{\perp}(x)\}, \max\{\mu_A^{\top}(x), \mu_B^{\top}(x)\}] \,|\, x \in X\}$,

$A \cap B = \{(x, [\min\{\mu_A^{\perp}(x), \mu_B^{\perp}(x)\}, \min\{\mu_A^{\top}(x), \mu_B^{\top}(x)\}] \,|\, x \in X\}$,

$A^c = \{(x, [1 - \mu_A^{\top}(x), 1 - \mu_A^{\perp}(x)]) | x \in X\}$,

where A^c is the *complement of interval valued fuzzy set* A in X.

3 Interval valued $(\in, \in \vee\, q)$-fuzzy filters

Based on the results of [1] and [2], we can extend the concept of quasi-coincidence of fuzzy point within a fuzzy set to the concept of quasi-coincidence of a fuzzy interval value with an interval valued fuzzy set.

An interval valued fuzzy set F of a pseudo BL-algebra A of the form

$$\widehat{\mu_F}(y) = \begin{cases} \widehat{t} \neq [0,0] & \text{if} \quad y = x, \\ [0,0] & \text{if} \quad y \neq x, \end{cases}$$

is said to be a *fuzzy interval value with support* x *and interval value* $\widehat{t}$ and is denoted by $U(x; \widehat{t})$. We say that a fuzzy interval value $U(x; \widehat{t})$ *belongs to* (or resp. is *quasi-coincident with*) an interval valued fuzzy set F, written by $U(x; \widehat{t}) \in F$ (resp. $U(x; \widehat{t})qF$) if $\widehat{\mu_F}(x) \geq \widehat{t}$ (resp. $\widehat{\mu_F}(x) + \widehat{t} > [1, 1]$). If $U(x; \widehat{t}) \in F$ or $U(x; \widehat{t})qF$, then we write $U(x; \widehat{t}) \in \vee\, q$. If $U(x; \widehat{t}) \in F$ and $U(x; \widehat{t})qF$, then we write $U(x; \widehat{t}) \in \wedge qF$. The symbol $\overline{\in \vee\, q}$ means that $\in \vee\, q$ does not hold.

In what follows, A is a pseudo BL-algebra unless otherwise specified. We emphasis that $\widehat{\mu_F}(x) = [\mu_F^{\perp}(x), \mu_F^{\top}(x)]$ must satisfy the following properties:

$[\mu_F^{\perp}(x), \mu_F^{\top}(x)] < [0.5, 0.5]$ or $[0.5, 0.5] \leq [\mu_F^{\perp}(x), \mu_F^{\top}(x)]$, for all $x \in A$.

First, we can extend the concept of fuzzy filters to the concept of interval valued fuzzy filters of A as follows:

Definition 3.1. An interval valued fuzzy set F of A is said to be an *interval valued fuzzy filter* of A if the following two conditions hold:

(F_1) $\widehat{\mu_F}(x \odot y) \geq \mathrm{rmin}\{\widehat{\mu_F}(x), \widehat{\mu_F}(y)\}$ $\forall x, y \in A$,

(F_2) $x \leq y \Rightarrow \widehat{\mu_F}(x) \leq \widehat{\mu_F}(y)$ $\forall x, y \in A$.

Let F be an interval valued fuzzy set. Then, for every $t \in (0,1]$, the set $F_{\widehat{t}} = \{x \in A \,|\, \widehat{\mu_F}(x) \geq \widehat{t}\}$ is called the *level subset of* F.

Now, we characterize the interval valued fuzzy filters by using their level filters.

Theorem 3.2. *An interval valued fuzzy set F of A is an interval valued fuzzy filter of A if and only if for any $[0,0] < \widehat{t} \leq [1,1]$ each non-empty $F_{\widehat{t}}$ is a filter of A.*

Proof. The proof is similar to Theorem 2.4. □

Further, we define the following concept:

Definition 3.3. An interval valued fuzzy set F of A is said to be an *interval valued $(\in, \in\vee q)$-fuzzy filter* of A if for all $t, r \in (0,1]$ and $x, y \in A$,

(F_3) $U(x; \widehat{t}\,) \in F$ and $U(y; \widehat{r}) \in F$ imply $U(x \odot y; \mathrm{rmin}\{\widehat{t}, \widehat{r}\}) \in\vee\, qF$,

(F_4) $U(x; \widehat{r}) \in F$ implies $U(y; \widehat{r}) \in\vee\, qF$ with $x \leq y$.

Example 3.4. Let I be a filter of a pseudo BL-algebra A and let F be an interval valued fuzzy set in A defined by

$$\widehat{\mu_F}(x) = \begin{cases} [0.7,0.8] & \text{if } x \in I, \\ [0.3,0.4] & \text{otherwsie.} \end{cases}$$

It is easily to verify that F is an interval valued $(\in, \in\vee q)$-fuzzy filter of A.

Theorem 3.5. *An interval valued fuzzy set F of A is an interval valued $(\in, \in\vee q)$-fuzzy filter if and only if for all $x, y \in A$ the following two conditions are satisfied:*

(F_5) $\widehat{\mu_F}(x \odot y) \geq \mathrm{rmin}\{\widehat{\mu_F}(x), \widehat{\mu_F}(y),\, 0.5\}$,

(F_6) $x \leq y \Longrightarrow \widehat{\mu_F}(y) \geq \mathrm{rmin}\{\widehat{\mu_F}(x),\, 0.5\}$.

Proof. At first we prove that the conditions (F_3) and (F_5) are equivalent.

Suppose that (F_3) do not implies (F_5), i.e., (F_3) holds but (F_5) is not satisfied. In this case there are $x, y \in A$ such that

$$\widehat{\mu_F}(x \odot y) < \mathrm{rmin}\{\widehat{\mu_F}(x), \widehat{\mu_F}(y), [0.5, 0.5]\}.$$

If $\mathrm{rmin}\{\widehat{\mu_F}(x), \widehat{\mu_F}(y)\} < [0.5, 0.5]$, then $\widehat{\mu_F}(x \odot y) < \mathrm{rmin}\{\widehat{\mu_F}(x), \widehat{\mu_F}(y)\}$. This means that for some t satisfying the condition $\widehat{\mu_F}(x \odot y) < \widehat{t} < \mathrm{rmin}\{\widehat{\mu_F}(x), \widehat{\mu_F}(y)\}$, we have $U(x; \widehat{t}\,) \in F$ and $U(y; \widehat{t}\,) \in F$, but $U(x \odot y; \widehat{t}\,) \overline{\in\vee q} F$, which contradicts to ($F_3$). So, this case is impossible. Therefore $\mathrm{rmin}\{\widehat{\mu_F}(x), \widehat{\mu_F}(y)\} \geq [0.5, 0.5]$. In this case $\widehat{\mu_F}(x \odot y) < [0.5, 0.5]$, $U(x; [0.5, 0.5]) \in F$, $U(y; [0.5, 0.5]) \in F$ and $U(x \odot y; [0.5, 0.5]) \overline{\in\vee q} F$, which also is impossible. So, (F_3) implies (F_5).

Conversely, if (F_5) holds and $U(x;\widehat{t}) \in F$, $U(y;\widehat{r}) \in F$, then $\widehat{\mu_F}(x) \geq \widehat{t}$, $\widehat{\mu_F}(y) \geq \widehat{r}$ and $\widehat{\mu_F}(x \odot y) \geq \mathrm{rmin}\{\widehat{t}, \widehat{r}, [0.5, 0.5]\}$. If $\mathrm{rmin}\{\widehat{t}, \widehat{r}\} > [0.5, 0.5]$, then $\widehat{\mu_F}(x \odot y) \geq [0.5, 0.5]$, which implies $F(x \odot y) + \mathrm{rmin}\{\widehat{t}, \widehat{r}\} > [1, 1]$, i.e., $U(x \odot y; \mathrm{rmin}\{\widehat{t}, \widehat{r}\}) qF$. If $\mathrm{rmin}\{\widehat{t}, \widehat{r}\} \leq [0.5, 0.5]$, then $\widehat{\mu_F}(x \odot y) \geq \mathrm{rmin}\{\widehat{t}, \widehat{r}\}$. Thus $U(x \odot y; \mathrm{rmin}\{\widehat{t}, \widehat{r}\}) \in F$. Therefore, $U(x \oplus y; \mathrm{rmin}\{\widehat{t}, \widehat{r}\}) \in \vee\, qF$. Summarizing, (F_5) implies (F_3). So (F_3) and (F_5) are equivalent.

To prove that (F_4) and (F_6) are equivalent suppose that (F_6) is not satisfied, i.e., $\widehat{\mu_F}(y_0) < \mathrm{rmin}\{\widehat{\mu_F}(x_0), [0.5, 0.5]\}$ for some $x_0 \leq y_0$. If $\widehat{\mu_F}(x_0) < [0.5, 0.5]$, then $\widehat{\mu_F}(y_0) < \widehat{\mu_F}(x_0)$, which means that there exists s such that $\widehat{\mu_F}(y_0) < \widehat{s} < \widehat{\mu_F}(x_0)$ and $\widehat{\mu_F}(y_0) + \widehat{\mu_F}(x_0) < [1, 1]$. Thus $U(y_0; \widehat{s}) \in F$ and $U(x_0; \widehat{s}) \overline{\in \vee\, q} F$, which contradicts to (F_4). So, $\widehat{\mu_F}(x_0) \geq [0.5, 0.5]$. But in this case $\widehat{\mu_F}(y_0) < \mathrm{rmin}\{\widehat{\mu_F}(x_0), [0.5, 0.5]\}$ gives $U(x_0; [0.5, 0.5]) \in F$ and $U(x_0; [0.5, 0.5]) \overline{\in \vee q} F$, which contradicts to (F_4). Hence $\widehat{\mu_F}(y) \geq \mathrm{rmin}\{\widehat{\mu_F}(x), [0.5, 0.5]\}$ for all $x \leq y$, i.e., (F_4) implies (F_6).

Conversely, if (F_6) holds, then $x \leq y$ and $U(x;\widehat{t}) \in F$ imply $\widehat{\mu_F}(x) \geq \widehat{t}$, and so $\widehat{\mu_F}(y) \geq \mathrm{rmin}\{\widehat{\mu_F}(x), [0.5, 0.5]\} \geq \mathrm{rmin}\{\widehat{t}, [0.5, 0.5]\}$. Thus $\widehat{\mu_F}(y) \geq \widehat{t}$ or $\widehat{\mu_F}(y) \geq [0.5, 0.5]$, according to $\widehat{t} \leq [0.5, 0.5]$ or $\widehat{t} > [0.5, 0.5]$. Therefore, $U(y;\widehat{t}) \in \vee\, qF$. Hence (F_6) implies (F_4). □

Proposition 3.6. *An interval valued fuzzy set F of A is an interval valued $(\in, \in \vee\, q)$-fuzzy filter if and only if*

(F_7) $\widehat{\mu_F}(1) \geq \mathrm{rmin}\{\widehat{\mu_F}(x), [0.5, 0.5]\}$ *holds for all* $x \in A$

and one of the conditions:

(F_8) $\widehat{\mu_F}(y) \geq \mathrm{rmin}\{\widehat{\mu_F}(x), \widehat{\mu_F}(x \to y), [0.5, 0.5]\}$,

(F_8') $\widehat{\mu_F}(y) \geq \mathrm{rmin}\{\widehat{\mu_F}(x), \widehat{\mu_F}(x \hookrightarrow y), [0.5, 0.5]\}$

is satisfied for all $x, y \in A$.

Proof. The proof is similar to the proof of Proposition 4.7 from the first part of [8], so we omit it. □

Now, we characterize the interval valued $(\in, \in \vee\, q)$-fuzzy filters by using their level subsets.

Theorem 3.7. *An interval valued fuzzy set F of A is an interval valued $(\in, \in \vee\, q)$-fuzzy filter if and only if for all $[0, 0] < \widehat{t} \leq [0.5, 0.5]$ all nonempty level subsets $F_{\widehat{t}}$ are filters of A.*

Proof. Let F be an interval valued $(\in, \in \vee\, q)$-fuzzy filter of A and $[0, 0] < \widehat{t} \leq [0.5, 0.5]$. If $x, y \in F_{\widehat{t}}$, then $\widehat{\mu_F}(x) \geq \widehat{t}$ and $\widehat{\mu_F}(y) \geq \widehat{t}$. Now we have $\widehat{\mu_F}(x \odot y) \geq \mathrm{rmin}\{\widehat{\mu_F}(x), \widehat{\mu_F}(y), [0.5, 0.5]\} \geq \mathrm{rmin}\{\widehat{t}, [0.5, 0.5]\} = \widehat{t}$. This means that $x \odot y \in F_{\widehat{t}}$. Let $x, y \in A$ be such that $x \leq y$. If $x \in F_{\widehat{t}}$, then, by (F_4), we have $\widehat{\mu_F}(y) \geq \mathrm{rmin}\{\widehat{\mu_F}(x), [0.5, 0.5]\} \geq \mathrm{rmin}\{\widehat{t}, [0.5, 0.5]\} = \widehat{t}$, which implies $y \in F_{\widehat{t}}$. Hence, $F_{\widehat{t}}$ is a filter of A.

Conversely, let F be an interval valued fuzzy set of A such that all nonempty $F_{\widehat{t}}$, where $[0,0] < \widehat{t} \leq [0.5, 0.5]$, are filters of A. Then, for every $x, y \in A$, we have

$\widehat{\mu_F}(x) \geq \text{rmin}\{\widehat{\mu_F}(x), \widehat{\mu_F}(y), [0.5, 0.5]\} = \widehat{t}_0$,

$\widehat{\mu_F}(y) \geq \text{rmin}\{\widehat{\mu_F}(x), \widehat{\mu_F}(y), [0.5, 0.5]\} = \widehat{t}_0$.

Thus, $x, y \in F_{\widehat{t}_0}$, and so $x \odot y \in F_{\widehat{t}_0}$, i.e., $\widehat{\mu_F}(x \odot y) \geq \text{rmin}\{\widehat{\mu_F}(x), \widehat{\mu_F}(y), [0.5, 0.5]\}$. If $x, y \in A$ and $x \leq y$, then $\widehat{\mu_F}(x) \geq \text{rmin}\{\widehat{\mu_F}(x), [0.5, 0.5]\} = \widehat{s}_0$. Hence $x \in F_{\widehat{s}_0}$, and so $y \in F_{\widehat{s}_0}$. Thus $\widehat{\mu_F}(y) \geq \widehat{s}_0 = \text{rmin}\{\widehat{\mu_F}(x), [0.5, 0.5]\}$. Therefore, F is an interval valued $(\in, \in\vee q)$-fuzzy filter of A. □

Naturally, we can establish a similar result when each nonempty $F_{\widehat{t}}$ is a filter of A for $[0.5, 0.5] < \widehat{t} \leq [1, 1]$.

Theorem 3.8. *For $[0.5, 0.5] < \widehat{t} \leq [1, 1]$ each nonempty level subset $F_{\widehat{t}}$ of an interval valued fuzzy set F of A is a filter if and only if for all $x, y \in A$ the following two conditions are satisfied:*

(F_9) $\text{rmax}\{\widehat{\mu_F}(x \odot y), [0.5, 0.5]\} \geq \text{rmin}\{\widehat{\mu_F}(x), \widehat{\mu_F}(y)\}$,

(F_{10}) $\text{rmax}\{\widehat{\mu_F}(y), [0.5, 0.5]\} \geq \widehat{\mu_F}(x)$ *when* $x \leq y$.

Proof. Let $F_{\widehat{t}}$ be a nonempty level subset of F. Assume that $F_{\widehat{t}}$ is a filter of A. If $\text{rmax}\{\widehat{\mu_F}(x \odot y), [0.5, 0.5]\} < \text{rmin}\{\widehat{\mu_F}(x), \widehat{\mu_F}(y)\} = \widehat{t}$ for some $x, y \in A$, then $[0.5, 0.5] < \widehat{t} \leq [1, 1]$, $\widehat{\mu_F}(x \odot y) < \widehat{t}$ and $x, y \in F_{\widehat{t}}$. Thus $x \odot y \in F_{\widehat{t}}$, whence $\widehat{\mu_F}(x \odot y) \geq \widehat{t}$, which contradicts to $\widehat{\mu_F}(x \odot y) < \widehat{t}$. So, (F_9) is satisfied.

If there exist $x, y \in A$ such that $\text{rmax}\{\widehat{\mu_F}(y), [0.5, 0.5]\} < \widehat{\mu_F}(x) = \widehat{t}$, then $[0.5, 0.5] < \widehat{t} \leq [1, 1]$, $\widehat{\mu_F}(y) < \widehat{t}$ and $x \in F_{\widehat{t}}$. Since $x \leq y$ we also have $y \in F_{\widehat{t}}$. Thus $\widehat{\mu_F}(y) \geq \widehat{t}$, which is impossible. Therefore $\text{rmax}\{\widehat{\mu_F}(y), [0.5, 0.5]\} \geq \widehat{\mu_F}(x)$ for $x \leq y$.

Conversely, suppose that the conditions (F_9) and (F_{10}) are satisfied. In order to prove that for $[0.5, 0.5] < \widehat{t} \leq [1, 1]$ each nonempty level subset $F_{\widehat{t}}$ is a filter of A assume that $x, y \in F_{\widehat{t}}$. In this case $[0.5, 0.5] < \widehat{t} \leq \text{rmin}\{\widehat{\mu_F}(x), \widehat{\mu_F}(y)\} \leq \text{rmax}\{\widehat{\mu_F}(x \odot y), [0.5, 0.5]\} = \widehat{\mu_F}(x \odot y)$, which proves $x \odot y \in F_{\widehat{t}}$. If $x \leq y$ and $x \in F_{\widehat{t}}$, then $[0.5, 0.5] < \widehat{t} \leq \widehat{\mu_F}(x) \leq \text{rmax}\{\widehat{\mu_F}(y), [0.5, 0.5]\} = \widehat{\mu_F}(y)$, and so $y \in F_{\widehat{t}}$. This completes the proof. □

Let $J = \{t \in (0, 1] \mid F_{\widehat{t}} \neq \emptyset\}$, where F is an interval valued fuzzy set of A. For $J = (0, 1]$ F is an ordinary interval valued fuzzy filter of A (Theorem 3.2); for $J = (0, 0.5]$ it is an $(\in, \in\vee q)$-fuzzy filter of A (Theorem 3.7).

In [26], Yuan, Zhang and Ren gave the definition of a fuzzy subgroup with thresholds which is a generalization of Rosenfeld's fuzzy subgroup, and also Bhakat and Das's fuzzy subgroup. Based on the results of [26], we can extend the concept of a fuzzy subgroup with thresholds to the concept of an interval valued fuzzy filter with thresholds in the following way:

Definition 3.9. Let $[0,0] \le \widehat{\alpha} < \widehat{\beta} \le [1,1]$. An interval valued fuzzy set F of A is called an *interval valued fuzzy filter with thresholds* $(\widehat{\alpha}, \widehat{\beta})$ if for all $x, y \in A$, the following two conditions are satisfied:

(F_{11}) $\text{rmax}\{\widehat{\mu_F}(x \odot y), \widehat{\alpha}\} \ge \text{rmin}\{\widehat{\mu_F}(x), \widehat{\mu_F}(y), \widehat{\beta}\}$,

(F_{12}) $\text{rmax}\{\widehat{\mu_F}(y), \widehat{\alpha}\} \ge \text{rmin}\{\widehat{\mu_F}(x), \widehat{\beta}\}$ for $x \le y$.

Theorem 3.10. *An interval valued fuzzy set F of A is an interval valued fuzzy filter with thresholds $(\widehat{\alpha}, \widehat{\beta})$ if and only if each nonempty $F_{\widehat{t}}$, where $\widehat{\alpha} < \widehat{t} \le \widehat{\beta}$ is a filter of A.*

Proof. The proof is similar to the proof of Theorems 3.7 and 3.8. □

4 Interval valued $(\in, \in \vee q)$-fuzzy implicative filters

Definition 4.1. An interval valued $(\in, \in \vee q)$-fuzzy filter F of A is called an *interval valued $(\in, \in \vee q)$-fuzzy implicative filter* if for all $x, y \in A$ it satisfies the condition:

$$(F_{13}) \quad \begin{cases} \widehat{\mu_F}(x) \ge \text{rmin}\{\widehat{\mu_F}((x \to y) \hookrightarrow x),\ [0.5, 0.5]\}, \\ \widehat{\mu_F}(x) \ge \text{rmin}\{\widehat{\mu_F}((x \hookrightarrow y) \to x),\ [0.5, 0.5]\}. \end{cases}$$

The following proposition is obvious.

Proposition 4.2. *If F is an interval valued $(\in, \in \vee q)$-fuzzy implicative filter of A, then*

$$(1) \quad \begin{cases} \widehat{\mu_F}(x) \ge \text{rmin}\{\widehat{\mu_F}((x \to y) \to x),\ [0.5, 0.5]\}, \\ \widehat{\mu_F}(x) \ge \text{rmin}\{\widehat{\mu_F}((x \hookrightarrow y) \hookrightarrow x),\ [0.5, 0.5]\}, \end{cases}$$

$$(2) \quad \begin{cases} \widehat{\mu_F}(((y \to x) \hookrightarrow x) \to y) \ge \text{rmin}\{\widehat{\mu_F}(x \to y),\ [0.5, 0.5]\}, \\ \widehat{\mu_F}(((y \hookrightarrow x) \to x) \hookrightarrow y) \ge \text{rmin}\{\widehat{\mu_F}(x \hookrightarrow y),\ [0.5, 0.5]\}, \end{cases}$$

$$(3) \quad \begin{cases} \widehat{\mu_F}((y \hookrightarrow x) \to x) \ge \text{rmin}\{\widehat{\mu_F}((x \to y) \hookrightarrow y),\ [0.5, 0.5]\}, \\ \widehat{\mu_F}((y \to x) \hookrightarrow x) \ge \text{rmin}\{\widehat{\mu_F}((x \hookrightarrow y) \to y),\ [0.5, 0.5]\}, \end{cases}$$

$$(4) \quad \begin{cases} \widehat{\mu_F}((y \to x) \hookrightarrow x) \ge \text{rmin}\{\widehat{\mu_F}((x \to y) \hookrightarrow y),\ [0.5, 0.5]\}, \\ \widehat{\mu_F}((y \hookrightarrow x) \to x) \ge \text{rmin}\{\widehat{\mu_F}((x \hookrightarrow y) \hookrightarrow y),\ [0.5, 0.5]\} \end{cases}$$

hold for all $x, y \in A$. □

Theorem 4.3. *An interval valued fuzzy set F of A is an interval valued $(\in, \in \vee q)$-fuzzy implicative filter if and only if for $[0,0] < \widehat{t} \le [0.5, 0.5]$ each nonempty level subset $F_{\widehat{t}}$ is an implicative filter of A.*

Proof. Let F be an interval valued $(\in, \in\vee q)$-fuzzy implicative filter of A and $[0,0] < \widehat{t} \leq [0.5, 0.5]$. Then, by Theorem 3.7, each nonempty $F_{\widehat{t}}$ is a filter of A. For all $x, y \in A$ from $(x \to y) \hookrightarrow x \in F_{\widehat{t}}$ it follows $\widehat{\mu_F}((x \to y) \hookrightarrow x) \geq \widehat{t}$. This, according to (F_{13}), gives $\widehat{\mu_F}(x) \geq \operatorname{rmin}\{\widehat{\mu_F}((x \to y) \hookrightarrow x), [0.5, 0.5]\} \geq \operatorname{rmin}\{\widehat{t}, [0.5, 0.5]\} = \widehat{t}$. So, $x \in F_{\widehat{t}}$. Analogously $(x \hookrightarrow y) \to x \in F_{\widehat{t}}$ implies $x \in F_{\widehat{t}}$. Therefore $F_{\widehat{t}}$ is an implicative filter of A.

Conversely, if F is an interval valued fuzzy set of A such that for $[0,0] < \widehat{t} \leq [0.5, 0.5]$ each nonempty level set $F_{\widehat{t}}$ is an implicative filter of A, then, by Theorem 3.7, F is an interval valued $(\in, \in\vee q)$-fuzzy filter of A. Putting $\widehat{\mu_F}((x \to y) \hookrightarrow x) \geq \widehat{s}_0 = \operatorname{rmin}\{\widehat{\mu_F}((x \to y) \hookrightarrow x), [0.5, 0.5]\}$, we obtain $(x \to y) \hookrightarrow x \in F_{\widehat{s}_0}$. Consequently, $x \in F_{\widehat{s}_0}$, i.e., $\widehat{\mu_F}(x) \geq \widehat{s}_0 = \operatorname{rmin}\{\widehat{\mu_F}((x \to y) \hookrightarrow x), [0.5, 0.5]\}$. This proves the first condition of (F_{13}). Similarly, we prove the second condition. □

Basing on our Theorem 3.8 the above result can be extended to the case $[0.5, 0.5] < \widehat{t} \leq [1, 1]$ in the following way:

Theorem 4.4. *For $[0.5, 0.5] < \widehat{t} \leq [1, 1]$ each nonempty level subset $F_{\widehat{t}}$ of an interval valued fuzzy set F of A is an implicative filter of A if and only if the conditions (F_9), (F_{10}) and*

$$(F_{14}) \quad \begin{cases} \operatorname{rmax}\{\widehat{\mu_F}(x),\ [0.5, 0.5]\} \geq \widehat{\mu_F}((x \to y) \hookrightarrow x), \\ \operatorname{rmax}\{\widehat{\mu_F}(x),\ [0.5, 0.5]\} \geq \widehat{\mu_F}((x \hookrightarrow y) \to x). \end{cases}$$

are satisfied for all $x, y \in A$.

Proof. According to Theorem 3.8, for $[0.5, 0.5] < \widehat{t} \leq [1, 1]$ each nonempty level subset $F_{\widehat{t}}$ of F is a filter of A if and only if F satisfies (F_9) and (F_{10}). So, we shall prove only that a filter $F_{\widehat{t}}$ is implicative if and only if F satisfies (F_{14}).

To prove (F_{14}) suppose the existence of $x, y \in A$ such that

$$\operatorname{rmax}\{\widehat{\mu_F}(x),\ [0.5, 0.5]\} < \widehat{t} = \widehat{\mu_F}((x \to y) \hookrightarrow x).$$

In this case $[0.5, 0.5] < \widehat{t} \leq [1, 1]$, $\widehat{\mu_F}(x) < \widehat{t}$ and $(x \to y) \hookrightarrow x \in F_{\widehat{t}}$. Since $F_{\widehat{t}}$ is an implicative filter of A, we have $x \in F_{\widehat{t}}$, and so $\widehat{\mu_F}(x) \geq \widehat{t}$, which is a contradiction. Similarly, we can prove the second inequality of (F_{12}).

Conversely, suppose that an interval valued fuzzy set F satisfies (F_{14}) and each nonempty $F_{\widehat{t}}$ is a filter of A. If $[0.5, 0.5] < \widehat{t} \leq [1, 1]$ and $(x \to y) \hookrightarrow x \in F_{\widehat{t}}$, then $[0.5, 0.5] < \widehat{t} \leq \operatorname{rmin}\{\widehat{\mu_F}((x \to y) \hookrightarrow x) \leq \operatorname{rmax}\{\widehat{\mu_F}(x), [0.5, 0.5]\} < \widehat{\mu_F}(x)$, which implies $x \in F_{\widehat{t}}$. Similarly, from $(x \hookrightarrow y) \to x \in F_{\widehat{t}}$ it follows $x \in F_{\widehat{t}}$. Thus, $F_{\widehat{t}}$ is an implicative filter of A. □

Basing on the method presented in [26], we can extend the concept of a fuzzy subgroup with thresholds to the concept of an interval valued fuzzy implicative filter with thresholds.

Definition 4.5. Let $[0,0] \leq \widehat{\alpha} < \widehat{\beta} \leq [1,1]$. An interval valued fuzzy set F of A is called an *interval valued fuzzy implicative filter with thresholds* $(\widehat{\alpha}, \widehat{\beta})$ of A if for all $x, y \in A$ it satisfies (F_{11}), (F_{12}) and

$$(F_{15}) \quad \begin{cases} \operatorname{rmax}\{\widehat{\mu_F}(x), \widehat{\alpha}\} \geq \operatorname{rmin}\{\widehat{\mu_F}((x \to y) \hookrightarrow x), \widehat{\beta}\}, \\ \operatorname{rmax}\{\widehat{\mu_F}(x), \widehat{\alpha}\} \geq \operatorname{rmin}\{\widehat{\mu_F}((x \hookrightarrow y) \to x), \widehat{\beta}\}. \end{cases}$$

Theorem 4.6. *An interval valued fuzzy set F of A is an interval valued fuzzy implicative filter with thresholds $(\widehat{\alpha}, \widehat{\beta})$ if and only if each nonempty $F_{\widehat{t}}$, where $\widehat{\alpha} < \widehat{t} \leq \widehat{\beta}$ is an implicative filter of A.*

Proof. The proof is similar to the proof of Theorems 4.3 and 4.4. ☐

Definition 4.7. An interval valued $(\in, \in\vee q)$-fuzzy filter of A is called an *interval valued $(\in, \in\vee q)$-fuzzy MV-filter* of A if

$$(F_{16}) \quad \begin{cases} \widehat{\mu_F}(((y \to x) \hookrightarrow x) \to y) \geq \operatorname{rmin}\{\widehat{\mu_F}(x \to y),\, [0.5, 0.5]\}, \\ \widehat{\mu_F}(((y \hookrightarrow x) \to x) \hookrightarrow y) \geq \operatorname{rmin}\{\widehat{\mu_F}(x \hookrightarrow y),\, [0.5, 0.5]\} \end{cases}$$

holds for all $x, y \in A$.

It follows from Proposition 4.2(2) that every interval valued $(\in, \in\vee q)$-fuzzy implicative filter is an interval valued $(\in, \in\vee q)$-fuzzy MV-filter.

Definition 4.8. An interval valued $(\in, \in\vee q)$-fuzzy filter of A is called an *interval valued $(\in, \in\vee q)$-fuzzy G-filter* of A if

$$(F_{17}) \quad \begin{cases} \widehat{\mu_F}(x \to y) \geq \operatorname{rmin}\{\widehat{\mu_F}(x \to (x \to y)),\, [0.5, 0.5]\}, \\ \widehat{\mu_F}(x \hookrightarrow y) \geq \operatorname{rmin}\{\widehat{\mu_F}(x \hookrightarrow (x \hookrightarrow y)),\, [0.5, 0.5]\} \end{cases}$$

holds for all $x, y \in A$.

Lemma 4.9. *Every interval valued $(\in, \in\vee q)$-fuzzy implicative filter is an interval valued $(\in, \in\vee q)$-fuzzy G-filter.*

Proof. Let F be an interval valued $(\in, \in\vee q)$-fuzzy implicative filter of A. As it is well know (see [8]), for any $x, y \in A$, we have $x \hookrightarrow (x \hookrightarrow y) \leq ((x \hookrightarrow y) \hookrightarrow y) \to (x \hookrightarrow y)$. Whence, according to (F_6) we obtain $\widehat{\mu_F}(((x \hookrightarrow y) \hookrightarrow y) \to (x \hookrightarrow y)) \geq \operatorname{rmin}\{\widehat{\mu_F}(x \hookrightarrow (x \hookrightarrow y)), [0.5, 0.5]\}$. From this, applying (F_{13}), we get $\widehat{\mu_F}(x \hookrightarrow y) \geq \operatorname{rmin}\{\widehat{\mu_F}(((x \hookrightarrow y) \hookrightarrow y) \to (x \hookrightarrow y)), [0.5, 0.5]\} \geq \operatorname{rmin}\{\widehat{\mu_F}(x \hookrightarrow (x \hookrightarrow y)), [0.5, 0.5]\}$. This proves the second inequality of (F_{17}).

The proof of the first inequality is similar. ☐

Theorem 4.10. *For any interval valued $(\in, \in\vee q)$-fuzzy filter F of A satisfying the identity $\widehat{\mu_F}(x \to y) = \widehat{\mu_F}(x \hookrightarrow y)$ the following two conditions are equivalent:*

(1) *F is an interval valued $(\in, \in\vee q)$-fuzzy implicative filter;*

(2) *F is an interval valued $(\in, \in\vee q)$-fuzzy MV-filter and an interval valued $(\in, \in\vee q)$-fuzzy G-filter.*

Proof. (1) $\Longrightarrow$ (2) By Proposition 4.2 (2) and Lemma 4.9.

(2) $\Longrightarrow$ (1) Let F be an interval valued $(\in, \in\vee q)$-fuzzy MV-filter and an interval valued $(\in, \in\vee q)$-fuzzy G-filter of A. From $(x \hookrightarrow y) \hookrightarrow x \leq (x \hookrightarrow y) \to ((x \hookrightarrow y) \hookrightarrow y)$ (see [8]), we have

$$\begin{aligned}\widehat{\mu_F}((x \hookrightarrow y) \hookrightarrow ((x \hookrightarrow y) \hookrightarrow y)) &= \widehat{\mu_F}((x \hookrightarrow y) \to ((x \hookrightarrow y) \hookrightarrow y)) \\ &\geq \mathrm{rmin}\{\widehat{\mu_F}((x \hookrightarrow y) \hookrightarrow x),\ [0.5, 0.5]\},\end{aligned}$$

which together with the fact that F is an interval valued $(\in, \in\vee q)$-fuzzy G-filter of A gives

$$\begin{aligned}\widehat{\mu_F}((x \hookrightarrow y) \hookrightarrow y) &\geq \mathrm{rmin}\{\widehat{\mu_F}((x \hookrightarrow y) \hookrightarrow ((x \to y) \hookrightarrow y)),\ [0.5, 0.5]\} \\ &\geq \mathrm{rmin}\{\widehat{\mu_F}((x \hookrightarrow y) \hookrightarrow x),\ [0.5, 0.5]\}.\end{aligned}$$

Moreover, from $y \leq x \to y$ we get $(x \hookrightarrow y) \hookrightarrow x \leq y \hookrightarrow x$, and consequently

$$\widehat{\mu_F}(y \hookrightarrow x) \geq \mathrm{rmin}\{F((x \hookrightarrow y) \hookrightarrow x),\ [0.5, 0.5]\}.$$

The fact that F is an interval valued $(\in, \in\vee q)$-fuzzy MV-filter of A implies

$$\begin{aligned}\widehat{\mu_F}(((x \hookrightarrow y) \to y) \hookrightarrow x) &\geq \mathrm{rmin}\{\widehat{\mu_F}(y \hookrightarrow x),\ [0.5, 0.5]\} \\ &\geq \mathrm{rmin}\{\widehat{\mu_F}((x \hookrightarrow y) \hookrightarrow x),\ [0.5, 0.5]\}.\end{aligned}$$

Since F is an interval valued $(\in, \in\vee q)$-fuzzy filter of A, we also have

$$\widehat{\mu_F}(x) \geq \mathrm{rmin}\{\widehat{\mu_F}(((x \hookrightarrow y) \to y) \hookrightarrow x),\ \widehat{\mu_F}((x \hookrightarrow y) \to y),\ [0.5, 0.5]\}.$$

Summarizing the above, we obtain $\widehat{\mu_F}(x) \geq \mathrm{rmin}\{\widehat{\mu_F}((x \hookrightarrow y) \hookrightarrow x),\ [0.5, 0.5]\}$. Hence, F is an interval valued $(\in, \in\vee q)$-fuzzy implicative filter of A. $\square$

Problem. Prove or disprove that in Theorem 4.10 the assumption $\widehat{\mu_F}(x \to y) = \widehat{\mu_F}(x \hookrightarrow y)$ is essential.

5 Implication-based interval valued fuzzy implicative filters

Fuzzy logic is an extension of set theoretic variables to terms of the linguistic variable truth. Some operators, like $\wedge, \vee, \neg, \to$ in fuzzy logic also can be defined by using the tables of valuations. Also, the extension principle can be used to derive definitions of the operators.

In the fuzzy logic, the truth value of fuzzy proposition P is denoted by $[P]$. The correspondence between fuzzy logical and set-theoretical notations is presented below:
$[x \in F] = F(x)$,
$[x \notin F] = 1 - F(x)$,
$[P \wedge Q] = \min\{[P], [Q]\}$,
$[P \vee Q] = \max\{[P], [Q]\}$,
$[P \to Q] = \min\{1, 1 - [P] + [Q]\}$,
$[\forall x P(x)] = \inf[P(x)]$,
$\models P \iff [P] = 1$ for all valuations.

Of course, various implication operators can be defined similarly. In the table presented below we give the example of such definitions. In this table α denotes the degree of truth (or degree of membership) of the premise, β is the values for the consequence, and I the result for the corresponding implication:

Name	Definition of Implication Operators
Early Zadeh	$I_m(\alpha,\beta) = \max\{1-\alpha, \min\{\alpha,\beta\}\}$,
Lukasiewicz	$I_a(\alpha,\beta) = \min\{1, 1-\alpha+\beta\}$,
Standard Star (Gödel)	$I_g(\alpha,\beta) = \begin{cases} 1 & \text{if} \quad \alpha \leq \beta, \\ \beta & \text{if} \quad \alpha > \beta, \end{cases}$
Contraposition of Gödel	$I_{cg}(\alpha,\beta) = \begin{cases} 1 & \text{if} \quad \alpha \leq \beta, \\ 1-\alpha & \text{if} \quad \alpha > \beta, \end{cases}$
Gaines–Rescher	$I_{gr}(\alpha,\beta) = \begin{cases} 1 & \text{if} \quad \alpha \leq \beta, \\ 0 & \text{if} \quad \alpha > \beta, \end{cases}$
Kleene–Dienes	$I_b(\alpha,\beta) = \max\{1-\alpha, \beta\}$.

The "quality" of these implication operators could be evaluated either by empirically or by axiomatically methods.

Below we consider the implication operator defined in the Lukasiewicz system of continuous-valued logic.

Definition 5.1. An interval valued fuzzy set F of A is called a *fuzzifying implicative filter* of A if for any $x, y, z \in A$ it satisfies the following four conditions:

(F_{18}) $\models \big[[x \in F] \wedge [y \in F] \to [x \odot y \in F]\big]$,

(F_{19}) $\models \big[\,[x \in F] \to [y \in F]\,\big]$ for any $x \leq y$,

(F_{20}) $\models \big[\,[(x \to y) \hookrightarrow x] \to [x \in F]\,\big]$,

(F_{21}) $\models \big[\,[(x \hookrightarrow y) \to x] \to [x \in F]\,\big]$.

The concept of the "standard" tautology can be generalized to the $\widehat{t}$-*tautology*, where $[0,0] < \widehat{t} \leq [1,1]$, in the following way:

$\models_{\widehat{t}} P \Longleftrightarrow [P] \geq \widehat{t}$ for all valuations.

This definition and results obtained in [26] gives for us the possibility to introduce such definition:

Definition 5.2. Let $[0,0] < \widehat{t} \leq [1,1]$ be fixed. An interval valued fuzzy set F of A is called a *t-implication-based interval valued fuzzy implicative filter* of A if for all $x, y, z \in A$ the following conditions hold:

(F_{22}) $\models_{\widehat{t}} \big[[x \in F] \wedge [y \in F] \rightarrow [x \odot y \in F]\big]$,

(F_{23}) $\models_{\widehat{t}} \big[[x \in F] \rightarrow [y \in F]\big]$ for all $x \leq y$,

(F_{24}) $\models_{\widehat{t}} \big[[(x \rightarrow y) \hookrightarrow x] \rightarrow [x \in F]]$,

(F_{25}) $\models_{\widehat{t}} \big[[(x \hookrightarrow y) \rightarrow x] \rightarrow [x \in F]\big]$.

In a special case when an implication operator is defined as I we obtain:

Corollary 5.3. *An interval valued fuzzy set F of A is a t-implication-based interval valued fuzzy implicative filter if and only if for all $x, y, z \in A$ it satisfies:*

(F_{26}) $I(\mathrm{rmin}\{\widehat{\mu_F}(x), \widehat{\mu_F}(y)\}, \widehat{\mu_F}(x \odot y)) \geq \widehat{t}$,

(F_{27}) $I(\mathrm{rmin}\{\widehat{\mu_F}(x), \widehat{\mu_F}(y)\}) \geq \widehat{t}$ *for all* $x \leq y$,

(F_{28}) $I(\mathrm{rmin}\{\widehat{\mu_F}((x \rightarrow y) \hookrightarrow x)\}, \widehat{\mu_F}(x)) \geq \widehat{t}$,

(F_{29}) $I(\mathrm{rmin}\{\widehat{\mu_F}((x \hookrightarrow y) \rightarrow x)\}, \widehat{\mu_F}(x)) \geq \widehat{t}$.

This gives a very good base for future study of filters in various algebraic systems with implication operators. As an example we present one theorem. In a similar way we can obtain other typical results.

Theorem 5.4. *Let F be an interval valued fuzzy set of A.*

(*i*) *If $I = I_{gr}$, then F is an* 0.5*-implication-based interval valued fuzzy implicative filter of A if and only if F is an interval valued fuzzy implicative filter with thresholds* $(\widehat{r} = [0,0], \widehat{s} = [1,1])$.

(*ii*) *If $I = I_g$, then F is an* 0.5*-implication-based fuzzy implicative filter of A if and only if F is an interval valued fuzzy implicative filter with thresholds* $(\widehat{r} = [0,0], \widehat{s} = [0.5, 0.5])$.

(*iii*) *If $I = I_{cg}$, then F is an* 0.5*-implication-based interval valued fuzzy implicative filter of A if and only if F is an interval valued fuzzy implicative filter with thresholds* $(\widehat{r} = [0.5, 0.5], \widehat{s} = [1,1])$.

Proof. The proofs are straightforward and hence are omitted. □

6 Conclusions

Interval valued fuzzy set theory emerges from the observation but that in a number of cases, no objective procedure is available for selecting the crisp membership degrees of elements in a fuzzy set. It was suggested to alleviate that problem by allowing to specify only an interval to which the actual membership degree is assumed to belong. In this paper, we considered different type of interval valued $(\in, \in\vee q)$-fuzzy filters of pseudo BL-algebras and investigated the relationship between these filters. Finally, we proposed the concept of implication-based interval valued fuzzy implicative filters of pseudo BL-algebras, which seems to be a good support for future study. The other direction of future study is an investigation of interval valued (α, β)-fuzzy (implicative) filters, where α, β are one of $\in, q, \in\vee q$ or $\in\wedge q$.

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